\# Koliko racionalnih članova ima u razvoju $(\sqrt[3]{2}+\sqrt[4]{3})^{100}$?

Rj: $$(\sqrt[3]{2}+\sqrt[4]{3})^{100} = \sum_{k=0}^{100}\binom{100}{k}(\sqrt[3]{2})^{100-k}(\sqrt[4]{3})^{k}$$
$$= \sum_{k=0}^{100}\binom{100}{k} 2^{\frac{100-k}{3}} 3^{\frac{k}{4}}$$

Da bi $3^{\frac{k}{4}}$ bio racionalan broj k mora biti djeljiv sa 4

tj. $k \in \{0, 4, 8, \dots, 92, 96, 100\}$ ...(*)

Da bi $2^{\frac{100-k}{3}}$ bio racionalan broj $100-k$ mora biti djeljiv sa 3. Drugim riječima $k \in \{1, 4, 7, 10, \dots, 16, \dots, 91, 94, 97, 100\}$.
...(**)

Iz (*) i (**) vidimo da binom ima racionalne članove za $k \in \{4, 16, 28, 40, 52, 64, 76, 88, 100\}$

II način:
$$(\sqrt[3]{2}+\sqrt[4]{3})^{100} = (2^{\frac{1}{3}}+3^{\frac{1}{4}})^{100} = \sum_{k=0}^{100}\binom{100}{k}(2^{\frac{1}{3}})^{100-k}(3^{\frac{1}{4}})^{k} = \sum_{k=0}^{100}\binom{100}{k}(2^{\frac{1}{3}})^{k}(3^{\frac{1}{4}})^{100-k}$$
$$= \sum_{k=0}^{100}\binom{100}{k} 2^{\frac{k}{3}} 3^{25-\frac{k}{4}}$$

k treba da bude djeljivo sa 3 i sa 4 tj. sa 12.

$k \in \{0, 12, 24, 36, 48, 60, 72, 84, 96\}$

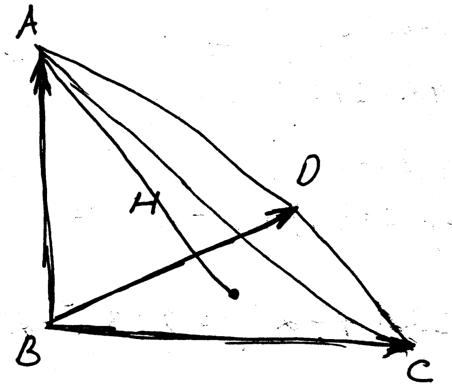
Postoji 9 racionalnih članova u razvoju binoma.

3. Date su tačke $A(3,2,1)$, $B(4,1,-2)$, $C(-5,-4,8)$ i $D(6,3,7)$. Odrediti:

a) zapreminu tetraedra $ABCD$.

b) visinu tetraedra koja odgovara osnovici BCD.

Rj.



$$\left.\begin{array}{l} B(4,1,-2) \\ A(3,2,1) \end{array}\right\} \Rightarrow \overrightarrow{BA}(-1,1,3)$$

$$D(6,3,7) \Rightarrow \overrightarrow{BD}(2,2,9)$$

$$C(-5,-4,8) \Rightarrow \overrightarrow{BC}(-9,-5,10)$$

a) $$V = \frac{1}{6}\left|(\overrightarrow{BC} \times \overrightarrow{BD}) \cdot \overrightarrow{BA}\right| = \frac{1}{6}\left|\begin{vmatrix} -9 & -5 & 10 \\ 2 & 2 & 9 \\ -1 & 1 & 3 \end{vmatrix}\right| \overset{I_k + II_k}{\underset{III_k - II_k \cdot 3}{=}} \frac{1}{6}\left|\begin{vmatrix} -14 & -5 & 25 \\ 4 & 2 & 3 \\ 0 & 1 & 0 \end{vmatrix}\right|$$

$$= \frac{1}{6}\left|\begin{vmatrix} -14 & 25 \\ 4 & 3 \end{vmatrix}\right| = \frac{1}{6}|-42-100| = \frac{142}{6} = \frac{71}{3}$$

Zapremina tetraedra $ABCD$ iznosi $\frac{71}{3}$.

b) Zapremina piramide $V = \frac{B \cdot H_{BCD}}{3}$

$$B = P_{\Delta ACD} = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BD}| = \frac{1}{2}\sqrt{4225 + 10\,201 + 64} = \frac{1}{2}\sqrt{9 \cdot 1610} = \frac{3}{2}\sqrt{1610}$$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -9 & -5 & 10 \\ 2 & 2 & 9 \end{vmatrix} = (-45-20)\vec{i} - (-81-20)\vec{j} + (-18+10)\vec{k}$$

$$= (-65, 101, -8)$$

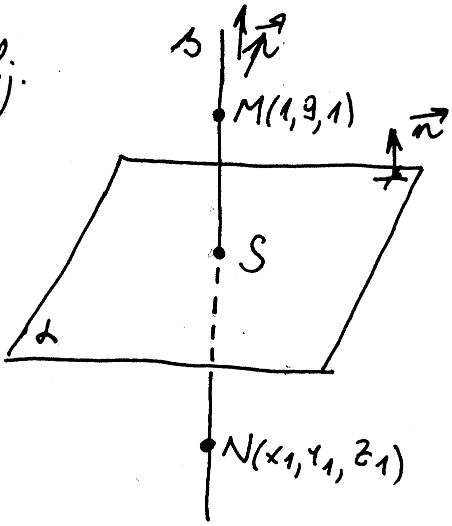
$$\frac{71}{3} = \frac{\frac{3}{2}\sqrt{1610} \cdot H_{BCD}}{3} \quad / \cdot 3 \cdot 2$$

$$3\sqrt{1610} \cdot H_{BCD} = 142$$

$$H_{BCD} = \frac{142}{3\sqrt{1610}}$$ je visina tetraedra koja odgovara osnovici BCD.

Odrediti tačku koja je simetrična tački $M(1,9,1)$ u odnosu na ravan $\alpha: 2x+y+3z=0$.

Rj.



$M(1,9,1)$

$\alpha: 2x+y+3z=0$

$M \notin \alpha$

$N=?$ $\quad |\overrightarrow{MS}| = |\overrightarrow{NS}|$

Da bismo odredili tačku N prvo ćemo postaviti pravu s koja je okomita na α i uz pomoć te prave naći tačku S.

$\vec{n} = (2,1,3)$

$\vec{p} \parallel \vec{n} \Rightarrow$ mogu uzeti $\vec{p} = (2,1,3)$ $\quad s: \dfrac{x-1}{2} = \dfrac{y-9}{1} = \dfrac{z-1}{3}$ $(=t)$

$$s: \begin{cases} x = 2t+1 \\ y = t+9 \\ z = 3t+1 \end{cases} \qquad \begin{aligned} x-1 &= 2t \\ y-9 &= t \\ z-1 &= 3t \end{aligned}$$

$2x+y+3z=0$

$2(2t+1)+(t+9)+3(3t+1)=0$

$4t+2+t+9+9t+3=0$

$14t=-14$

$t=-1$

Tačka presjeka prave s i ravni α je $S(-1,8,-2)$

$M(1,9,1)$
$S(-1,8,-2)$ $\quad \overrightarrow{MS} = (-2,-1,-3)$

$N(2t+1,\ t+9,\ 3t+1)$
$S(\ -1,\ \ 8,\ \ -2)$ $\quad \overrightarrow{NS} = (-2t-2,\ -t-1,\ -3t-3)$

$|\overrightarrow{MS}| = \sqrt{4+1+9} = \sqrt{14}$

$|\overrightarrow{NS}| = \sqrt{(-2t-2)^2+(-t-1)^2+(-3t-3)^2}$

$$\begin{aligned} (-2t-2)^2 &= 4t^2+8t+4 \\ (-t+1) &= t^2+2t+1 \\ (-3t-3)^2 &= 9t^2+18t+9 \\ \hline & 14t^2+28t+14 \end{aligned}$$

$|\overrightarrow{MS}| = |\overrightarrow{NS}|$

$14t^2+28t+14 = 14 \quad /:14$

$t^2+2t=0$

$t(t+2)=0$

$t=0$ ili $t=-2$

$N(-3,7,-5)$ tražena tačka

(#) Ispitati f-ju i nacrtati joj grafik $y=\dfrac{3x}{1+x^3}$.

Rj. definiciono područje

$1+x^3 \neq 0$

$x^3 \neq -1$

$x \neq -1$

$\mathcal{D}: x \in (-\infty,-1) \cup (-1,+\infty)$

parnost, neparnost, periodičnost

$f(-x)=\dfrac{3\cdot(-x)}{1+(-x^3)} = -\dfrac{3x}{1-x^3}$ f-ja nije ni parna ni neparna

f-ja nije periodična

nule, presjek sa y-osom, znak f-je

$y=0$ (0,0) je nula f-je i presjek sa y-osom

$\dfrac{3x}{1+x^3}=0$

$x=0$

x	$(-\infty,-1)$	$(-1,0)$	$(0,+\infty)$
$3x$	$-$	$-$	$+$
$1+x^3$	$-$	$+$	$+$
y	$+$	$-$	$+$

znak f-je

$-1 \quad 0 \quad 1$

ponašanje na krajevima intervala definisanosti i asimptote

za vrijednost $x=-1$ f-ja ima prekid

$\lim\limits_{x\to -1-0} f(x) = \lim\limits_{x\to -1-0} \dfrac{3x}{1+x^3} = \dfrac{3(-1-0)}{1+(-1-0)^3} = \dfrac{3(-1-0)}{1-1-0} = \dfrac{-3-0}{-0} = +\infty \Rightarrow x=-1$ je V.A.

$\lim\limits_{x\to -1+0} f(x) = \lim\limits_{x\to -1+0} \dfrac{3x}{1+x^3} = \dfrac{3(-1+0)}{1+(-1+0)^3} = \dfrac{-3+0}{1-1+0} = \dfrac{-3+0}{+0} = -\infty \Rightarrow x=-1$ je V.A.

$\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} \dfrac{3x \;:x}{1+x^3 \;:x} = \lim\limits_{x\to -\infty} \dfrac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2} = 0 \Rightarrow y=0$ je H.A.

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} \dfrac{3}{\underbrace{\frac{1}{x}}_{\to 0}+x^2} = 0 \Rightarrow y=0$ je H.A. f-ja nema K.A.

rast i opadanje

$y' = \left(\dfrac{3x}{1+x^3}\right)' = 3\cdot \dfrac{1\cdot(1+x^3)-x\cdot 3x^2}{(1+x^3)^2} = 3\dfrac{1+x^3-3x^3}{(1+x^3)^2}$

$y' = 3\cdot \dfrac{1-2x^3}{(1+x^3)^2}$

$y'=0$ akko $1-2x^3=0$

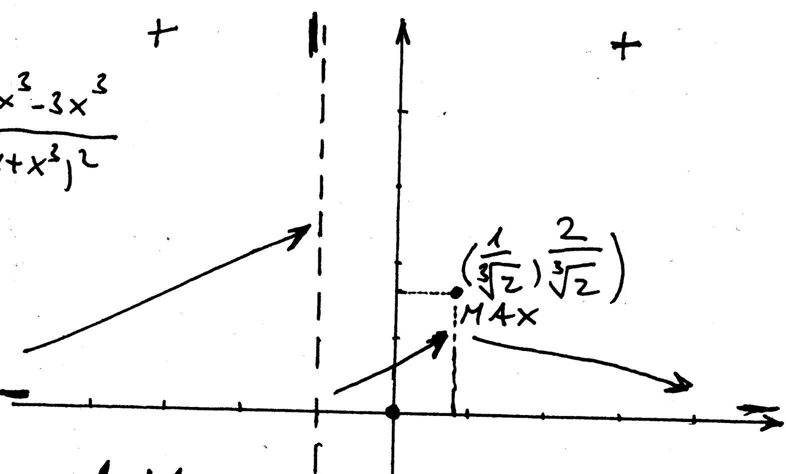
$2x^3=1$

$x^3=\frac{1}{2}$

$x=\frac{1}{\sqrt[3]{2}} \approx 0{,}8$

$-1 \quad \frac{1}{\sqrt[3]{2}}$

prekidi y + nule y'



x	$(-\infty,-1)$	$(-1,\frac{1}{\sqrt[3]{2}})$	$(\frac{1}{\sqrt[3]{2}},+\infty)$
y'	$+$	$+$	$-$
y	↗	↗	↘

MAX

ekstrem f-je

Na osnovu tabele

$f\left(\frac{1}{\sqrt[3]{2}}\right) = \dfrac{\frac{3}{\sqrt[3]{2}}}{1+\frac{1}{2}} = \dfrac{\frac{3}{\sqrt[3]{2}}}{\frac{3}{2}} = \dfrac{2}{\sqrt[3]{2}} \approx 1{,}6$

$\left(\frac{1}{\sqrt[3]{2}}, \frac{2}{\sqrt[3]{2}}\right)$ je tačka maksimuma prevojne tačke i intervali konveksnosti i konkavnosti

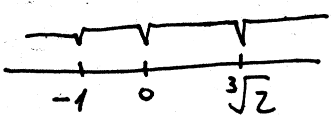
$$y' = 3 \cdot \frac{1-2x^3}{(1+x^3)^2}, \quad y'' = 3 \cdot \frac{-6x^2 \cdot (1+x^3)^{\cancel{2}} - (1-2x^3)\cdot 2\cancel{(1+x^3)} \cdot 3x^2}{(1+x^3)^3 \cdot \cancel{(1+x^3)}} =$$

$$= 3 \cdot \frac{-6x^2 - 6x^5 - 6x^2 + 12x^5}{(1+x^3)^3} = 3 \cdot \frac{6x^5 - 12x^2}{(1+x^3)^3}$$

$$y'' = 18 \cdot \frac{x^5 - 2x^2}{(1+x^3)^3} = \frac{18x^2(x^3-2)}{(1+x^3)^3}$$

$y''=0$ akko $x=0$ ili $x^3-2=0$

$x_1 = 0 \qquad x_2 = \sqrt[3]{2} \approx 1{,}3$



x	$(-\infty,-1)$	$(-1,0)$	$(0,\sqrt[3]{2})$	$(\sqrt[3]{2},+\infty)$
y''	+	−	−	+
y	$\cup$	$\cap$	$\cap$	$\cup$

P_0T_0

$$f(\sqrt[3]{2}) = \frac{3\sqrt[3]{2}}{1+2} = \sqrt[3]{2}$$

$(\sqrt[3]{2}, \sqrt[3]{2})$ je prevojna tačka

grafik